In this section, we introduce the simplest applications of generalized unitarity for loop amplitudes.

Generalized unitarity is an idea to reconstruct loop amplitude *integrand* from tree amplitude. It originates from the research of Zvi Bern, Lance Dixon and David Kosower in 1990s. Since tree amplitude is simple (on-shell and ghost-free), soon this method became a very popular tool for modern quantum field theory. Different from the original unitarity in QFT (optical theorem), with generalized unitarity we can apply any cuts to an loop amplitude to reconstruct various terms in the integrand.

Note that generalized unitarity does not directly produce the *integrated* loop amplitudes. The loop integral integration would be introduced in the following sections.

I. OSSOLA-PAPADOPOULOS-PITTAU INTEGRAND REDUCTION

Ossola, Papadopoulos and Pittau (OPP) integrand reduction [1, 2] is a modern tool for analyzing the loop integrand of an *one-loop* amplitude. By partial fraction, it decompose the one-loop integrand to a few simple terms. The basic idea is that for an one-loop amplitude,

$$\int d^D l \frac{N}{D_1 \dots D_k} \tag{1}$$

We mathematically decompose the numerator as,

$$N = \Delta + f_1 D_1 + f_2 D_2 + \dots f_k D_k \tag{2}$$

Here Δ should be the "simplest" and "independent" of D_i's. The rigorous meaning of "simplest" and "independent" would be introduced in the future. Note that $f_i D_i$ cancels one denominator and only contributes to the sub-diagrams.

A. Maximal cut example

We start with the 4D toy-version of the OPP integrand reduction. For simple $D = 4$ cases, we only need to begin with the box diagram. For instance, consider $D = 4$ four-point massless box,

$$D_1 = l^2, \quad D_2 = (l - p_1)^2, \quad D_3 = (l - p_1 - p_2)^2, \quad D_4 = (l + p_4)^2. \tag{3}$$

The Mandelstam variables are $s = (p_1 + p_2)^2$ and $t = (p_1 + p_4)^2$. It is useful to re-parameterize the loop momentum l instead of using its Lorentz components. There are several parametrization methods: (1) van Neerven-Vermaseren parameterization (2) spinor-helicity parameterization.

To make a $4D$ basis, we introduce an auxiliary vector $\omega_\mu \equiv \frac{2i}{s}\epsilon_{\mu\nu\rho\sigma}p_1^\nu p_2^\rho p_4^\sigma$. Explicitly,

$$\omega = \frac{s+t}{s}\frac{\langle 43\rangle}{\langle 13\rangle}1\tilde{4} + \frac{\langle 13\rangle}{\langle 43\rangle}4\tilde{1} \tag{4}$$

with

$$\omega^2 = -\frac{t(s+t)}{s}\,. \tag{5}$$

Then the basis $\{e_1, e_2 \ldots e_4\} \equiv \{p_1, p_2, p_4, \omega\}$. The Gram matrix of this basis is,

$$G = \begin{pmatrix} 0 & \frac{s}{2} & \frac{t}{2} & 0 \\ \frac{s}{2} & 0 & \frac{-s-t}{2} & 0 \\ \frac{t}{2} & \frac{-s-t}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{t(s+t)}{s} \end{pmatrix}, \quad G_{ij} = e_i \cdot e_j\,. \tag{6}$$

Note that for any well-defined basis, Gram matrix should be non-degenerate. For any $4D$ momentum p, define van Neerven-Vermaseren variables as,

$$x_i(p) \equiv p \cdot e_i, \quad i = 1, \ldots, 4\,. \tag{7}$$

Then for any two $4D$ momenta, a scalar product translates to van Neerven-Vermaseren form, by linear algebra

$$p_1 \cdot p_2 = \mathbf{x}(p_1)^T (G^{-1})\mathbf{x}(p_2)\,, \tag{8}$$

where the bold $\mathbf{x}(p)$ denotes the column 4-vector, $(x_1, x_2, x_3, x_4)^T$. Back to our one-loop box, define $x_i \equiv x_i(l)$. Hence a Lorentz-invariant numerator N_{box} has the form,

$$N_{\text{box}} = \sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4} c_{m_1 m_2 m_3 m_4} x_1^{m_1} x_2^{m_2} x_3^{m_3} x_4^{m_4}\,, \tag{9}$$

For a renormalizable theory, there is a bound on the sum, $m_1 + m_2 + m_3 + m_4 \leq 4$. The goal in integrand reduction is to expand

$$N_{\text{box}} = \Delta_{\text{box}} + h_1 D_1 + \ldots h_4 D_4\,, \tag{10}$$

such that the remainder polynomial Δ_{box} is as simple as possible.

The simplest Δ_{box} can be obtained by a direct analysis. Note that

$$\begin{aligned} x_1 &= l \cdot p_1 = \frac{1}{2}(D_1 - D_2), \\ x_2 &= l \cdot p_2 = \frac{1}{2}(D_2 - D_3) + \frac{s}{2}, \\ x_3 &= l \cdot p_4 = \frac{1}{2}(D_4 - D_1), \end{aligned} \tag{11}$$

hence x_1, x_2 and x_3 can be written as combinations of D_i's. A scalar product which equals combinations of denominators and constants is called a *reducible scalar product* (RSP). In this cases, x_1, x_2, x_3 are RSPs. The remainder Δ_{box} shall not depend on RSPs, hence,

$$\Delta_{\text{box}} = \sum_{m_4} c_{m_4} x_4^{m_4}. \tag{12}$$

x_4 is called a *irreducible scalar product* (ISP). Furthermore, using the expansion of l^2 and (11),

$$\begin{aligned} D_1 = l_1^2 = \frac{1}{4st(s+t)}\big(& -4s^2x_4^2 + s^2t^2 + 4D_1s^2t - 2D_2s^2t - 2D_4s^2t + D_2^2s^2 + D_4^2s^2 \\ & - 2D_2D_4s^2 + 2D_1st^2 - 2D_3st^2 + 2D_1D_2st - 4D_1D_3st + 2D_2D_3st + 2D_1D_4st \\ & - 4D_2D_4st + 2D_3D_4st + D_1^2t^2 + D_3^2t^2 - 2D_1D_3t^2\big), \end{aligned} \tag{13}$$

which means

$$x_4^2 = \frac{t^2}{4} + \mathcal{O}(D_i). \tag{14}$$

Hence quadratic and higher-degree monomials in x_4 should be removed from the box integrand, and

$$\Delta_{\text{box}} = c_{\text{box},0} + c_{\text{box},1}(l \cdot \omega). \tag{15}$$

This is the *integrand basis* for the $4D$ box, which contains only 2 terms. Note that by Lorentz symmetry,

$$\int d^4l \frac{l \cdot \omega}{D_1D_2D_3D_4} = 0, \tag{16}$$

for any value of D. So c_1 should not appear in the final expression of scattering amplitude. We call such a term a *spurious term.* But it is important for integrand reduction, as we will see soon.

II. GENERALIZED UNITARITY FOR THE MAXIMAL CUT COEFFICIENTS

The power of OPP integrand reduction is that the coefficients in the OPP basis can be directly fitted from tree amplitudes.

A. Gluon amplitude $1^-2^-3^+4^+$

For example, let us consider the color-order one-loop Yang-Mills amplitude $A^{(1)}(1^-, 2^-, 3^+, 4^+)$. We consider the quadruple cut (4-cut)

$$D_1 = D_2 = D_3 = D_4 = 0\,. \tag{17}$$

This is a generalized unitarity cut, different from the optical theorem. We expect that with the quadruple cut, all propagators become on-shell and we have a product of four tree amplitudes (summed over all internal physical states):

$$N|_{4-cut} = \sum_{h_1,h_2,h_3,h_4=\pm} A_1(-l^{*,-h_1}, p_1, (l^* - p_1)^{h_2}) A_2((p_1 - l^*)^{-h_2}, p_2, (l^* - p_1 - p_2)^{h_3}) \times A_3((p_1 + p_2 - l^*)^{-h_3}, p_3, (l^* + p_4)^{h_4}) A_4((-l^* - p_4)^{-h_4}, p_4, l^{h_1}) \tag{18}$$

where l^* is a particular value such that

$$l^{*2} = 0, \quad (l^* - p_1)^2 = 0, \quad (l^* - p_1 - p_2)^2 = 0, \quad (l^* + p_4)^2 = 0. \tag{19}$$

Note that the (19) may have several solutions. For this quadruple cut case, we can parameterize the loop momenta as

$$l = a_1 p_1 + a_2 p_4 + a_3 \frac{\langle 43 \rangle}{\langle 13 \rangle} 1\tilde{4} + a_4 \frac{\langle 13 \rangle}{\langle 43 \rangle} 4\tilde{1} \tag{20}$$

It is easy to see that there are two solutions

$$\text{(I)}: \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = -1, \quad a_4 = 0 \tag{21}$$

$$\text{(II)}: \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = s/(s+t) \tag{22}$$

Note that the second term in the box OPP basis, $l \cdot \omega$, has the values at the two solutions

$$\text{(I)}: \quad l \cdot \omega \to \frac{t}{2} \tag{23}$$

$$\text{(II)}: \quad l \cdot \omega \to -\frac{t}{2} \tag{24}$$

The correspondence between the two OPP coefficients and the two solutions can be illustrated by the following examples.

B. $A^{(1)}(1^-2^-3^+4^+)$

This is a easy case for generalized unitarity.

- For the solution (I), note that with this solution $l^*, l^* - p_1, l^* - p_1 - p_2, l^* + p_4$ are all null vectors. Therefore, we split them into spinors tables,

l^*	$x1$	$\tilde{4}$
$l^* - p_1$	1	$x\tilde{4} - \tilde{1}$
$l^* - p_1 - p_2$	$y1 - 2$	$\tilde{2}$
$l^* + p_4$	$x1 + 4$	$\tilde{4}$

(25)

From the spinor table, we see that A_1, A_3 must be $\overline{\text{MHV}}$, and A_2, A_4 must be MHV. Here

$$x = -\frac{\langle 43\rangle}{\langle 13\rangle}, \quad y = -\frac{[14]}{[24]} \tag{26}$$

Furthermore, we can see that there is only one internal helicity configuration for this solution. We get

$$\sum_{\text{helicities}} A_1 A_2 A_3 A_4|_{\text{solution (I)}} = istA(1^- 2^- 3^+ 4^+)\,. \tag{27}$$

- For the solution (II), we see the spinor table is

$$\begin{array}{c|c|c} l^* & x4 & \tilde{1} \\ \hline l^* - p_1 & x4 - 1 & \tilde{1} \\ \hline l^* - p_1 - p_2 & 2 & y\tilde{1} - \tilde{2} \\ \hline l^* + p_4 & 4 & x\tilde{1} + \tilde{4} \end{array} \tag{28}$$

From the spinor table, we see that A_1, A_3 must be MHV, and A_2, A_4 must be $\overline{\text{MHV}}$. Here

$$x = -\frac{s\langle 13\rangle}{(s+t)\langle 34\rangle}, \quad y = -\frac{t\langle 13\rangle}{(s+t)\langle 23\rangle} \tag{29}$$

Furthermore, we can see that there is only one internal helicity configuration for this solution. We get

$$\sum_{\text{helicities}} A_1 A_2 A_3 A_4|_{\text{solution (II)}} = istA(1^- 2^- 3^+ 4^+)\,. \tag{30}$$

Combine the two cases together, we see that

$$c_{\text{box},0}^{--++} = istA(1^- 2^- 3^+ 4^+), \quad c_{\text{box},1}^{--++} = 0 \tag{31}$$

C. Gluon amplitude $1^- 2^+ 3^- 4^+$

The previous example is somehow deceptive in the sense that the two cuts gives the same tree product, via general unitarity. In the one-loop gluon amplitude $1^- 2^+ 3^- 4^+$ example, we see that the feature is absent.

- For the solution (I), again we split the loop momenta into spinors tables. It is easy to see that there is only one helicity configuration and,

$$d_1 \equiv \sum_{\text{helicities}} A_1 A_2 A_3 A_4|_{\text{solution (I)}} = istA(1^- 2^+ 3^- 4^+)\,. \tag{32}$$

- For the solution (II), we see that there are two helicity configurations and the sum is that

$$d_2 \equiv \sum_{\text{helicities}} A_1 A_2 A_3 A_4|_{\text{solution (II)}} = ist\left(\frac{s^4}{(s+t)^4} + \frac{t^4}{(s+t)^4}\right) A(1^- 2^+ 3^- 4^+)\,. \tag{33}$$

The OPP coefficients can be determined as,

$$c_{\text{box},0} + c_{\text{box},1}\frac{t}{2} = d_1 \tag{34}$$

$$c_{\text{box},0} - c_{\text{box},1}\frac{t}{2} = d_2 \tag{35}$$

Then,

$$\begin{aligned} c_{\text{box},0}^{-+-+} &= \frac{1}{2}\left(d_1 + d_2\right) = i\frac{st(s^2+st+t^2)^2}{(s+t)^4} A(1^- 2^+ 3^- 4^+) \\ c_{\text{box},1}^{-+-+} &= \frac{1}{t}\left(d_1 - d_2\right) = i\frac{2s^2 t(2s^2+3st+2t^2)}{(s+t)^4} A(1^- 2^+ 3^- 4^+) \end{aligned} \tag{36}$$

Note that in the $N=4$ super-Yang-Mills theory, with the contribution from gluino and scalars, the two cuts both give the same tree products $istA(1^- 2^+ 3^- 4^+)$. Therefore

$$\begin{aligned} c_{\text{box},0}^{\text{SYM},-+-+} &= istA(1^- 2^+ 3^- 4^+) \\ c_{\text{box},1}^{\text{SYM},-+-+} &= 0 \end{aligned} \tag{37}$$

So we see that the $N=4$ integrand is significantly simpler than the corresponding QCD case.

D. Gluon amplitude $A^{(1)}(1^- 2^- 3^+ 4^+ 5^+)$

We consider the five-point one-loop amplitude in Yang-Mills theory, $A^{(1)}(1^- 2^- 3^+ 4^+ 5^+)$.

First question, can we use the simple but less rigorous 4 dimensional OPP reduction and 4 dimensional generalized unitarity? The answer is "yes", in some sense.

Note that in principle there is a pentagon-box diagram for this one-loop amplitude. The pentagon part cannot be determined by the 4-dimensional generalized unitarity. (There are four components for the loop momenta but five cut equations.) However, in the rigorous D dimensional unitarity approach, we will see that the "genuine" pentagon part provides only a vanishing term (in the high order of ϵ). We postpone the D-dimensional discussion for a while and then use the 4 dimensional generalized unitarity method to determine the box coefficient.

There would be five different boxes for this amplitude. Here we determine the box coefficient for a particular box with the external states 1 and 2 combined together, $\text{box}_{12;345}$.

The propagators on the quadruple cut are

$$D_1 = l^2, \quad D_3 = (l - p_1 - p_2)^2, \quad D_4 = (l - p_1 - p_2 - p_3)^2, \quad D_5 = (l + p_5)^2 \tag{38}$$

The first solution has the spinor helicity table

$$\begin{array}{c|c|c} l^* & 5 & x\tilde{4} - \tilde{5} \\ \hline l^* - p_1 - p_2 & 3 & y\tilde{4} + \tilde{3} \\ \hline l^* - p_1 - p_2 - p_3 & 4 + 5x & \tilde{4} \\ \hline l^* + p_4 & 5 & x\tilde{4} \end{array} \tag{39}$$

where $x = -\langle 43\rangle/\langle 53\rangle$ and $y = \langle 45\rangle/\langle 35\rangle$. There is only one helicity configuration and the tree product gives

$$d_1 = \sum_{\text{helicities}} A_1 A_2 A_3 A_4|_{\text{solution (I)}} = i s_{34} s_{45} A(1^- 2^- 3^+ 4^+ 5^+)\,. \tag{40}$$

The second solution provides the zero tree product

$$d_2 = \sum_{\text{helicities}} A_1 A_2 A_3 A_4|_{\text{solution (II)}} = 0. \tag{41}$$

Combine them together we find that

$$c^{--+++}_{\text{box},12;345,0} = \frac{i s_{34} s_{45}}{2} A(1^- 2^- 3^+ 4^+ 5^+) \tag{42}$$

as well as a nonzero spurious coefficient.

III. NON-MAXIMAL CUTS

For the OPP-like approach, after determine the top integral coefficient from the maximal cuts, we need to determine the daughter integral coefficients from the non-maximal cuts.

Usually, it takes more efforts to apply non-maximal cuts and the computation is more involved.

A. Triple cuts

For the one-loop box diagram with propagators D_1, D_2, D_3 and D_4, there are four daughter diagram with three propagators each. Here we focus on the

$$\int d^D l \frac{N}{D_1 D_2 D_4} \tag{43}$$

triangle diagram. Again we first use the OPP reduction to determine the genuine numerators contributing to the triangle diagram. In the $4D$ OPP approach, we would like to split

$$N = \Delta_{\text{tri}} + f_1 D_1 + f_2 D_2 + f_4 D_4 \tag{44}$$

where f_i's are polynomials in the loop momentum. The goal is to move "as many as possible" terms in N to $f_1 D_1 + f_2 D_2 + f_4 D_4$, in order to get a "simplest" Δ_{tri}.

Again, we use van Neerven-Vermaseren parameterization. Define a vector basis p_1, p_4, v_1, v_2 with

$$v_1 = \frac{\langle 43\rangle}{\langle 13\rangle} 1\tilde{4} \tag{45}$$

$$v_2 = \frac{\langle 13\rangle}{\langle 43\rangle} 4\tilde{1} \tag{46}$$

Note that $p_1 \cdot v_i = 0$ and $p_4 \cdot v_i = 0$. We define the van Neerven-Vermaseren variables,

$$x_1 = l \cdot p_1, \quad x_2 = l \cdot p_4, \quad x_3 = l \cdot v_1, \quad x_4 = l \cdot v_2 \tag{47}$$

Then the propagators read

$$D_1 = \frac{4(x_1 x_2 - x_3 x_4)}{t}$$
$$D_2 = \frac{2(-t x_1 + 2 x_1 x_2 - 2 x_3 x_4)}{t}$$
$$D_3 = \frac{2(t x_2 + 2 x_1 x_2 - 2 x_3 x_4)}{t}$$

Then it is easy to see that $x_1 = (D_1 - D_2)/2$ and $x_2 = (D_3 - D_1)/2$. Furthermore, $x_3 x_4$ is a also a polynomial of D_i's. Hence, we have the OPP basis for this triangle

$$\Delta_{\text{tri}} = c_{\text{tri},0} + c_{\text{tri},1} x_3 + c_{\text{tri},2} x_3^2 + c_{\text{tri},3} x_3^3 + c_{\text{tri},4} x_4 + c_{\text{tri},5} x_4^2 + c_{\text{tri},6} x_4^3 \,. \tag{48}$$

where for the maximal power we used the renormalization constraint.

Note that by the Passarino-Veltman reduction, $c_{\text{tri},j}$, $j = 1, \ldots 6$ are all spurious.

The triple cut is to set $D_1 = D_2 = D_4 = 0$. We use the previous spinor helicity formalism for l,

$$l = a_1 p_1 + a_2 p_4 + a_3 \frac{\langle 43\rangle}{\langle 13\rangle} 1\tilde{4} + a_4 \frac{\langle 13\rangle}{\langle 43\rangle} 4\tilde{1} \tag{49}$$

Then there are two triple cut solution branches,

$$(\text{I}): \quad a_1 = 0, \quad a_2 = 0, \quad a_4 = 0 \tag{50}$$

$$(\text{II}): \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0 \tag{51}$$

Note that each solution is one-dimensional. The solution (I) has a free parameter a_3 and the solution (II) has a free parameter a_4.

The OPP triangle basis on the two solution reads

$$\begin{aligned}\Delta_{\text{tri}}|_{\text{solution-I}} &= c_{\text{tri},0} - \frac{tc_{\text{tri},4}}{2}a_3 + \frac{t^2c_{\text{tri},5}}{4}a_3^2 - \frac{t^3c_{\text{tri},6}}{8}a_3^3 \\ \Delta_{\text{tri}}|_{\text{solution-II}} &= c_{\text{tri},0} - \frac{tc_{\text{tri},1}}{2}a_4 + \frac{t^2c_{\text{tri},2}}{4}a_4^2 - \frac{t^3c_{\text{tri},3}}{8}a_4^3\end{aligned} \tag{52}$$

Note that they are all polynomials in the free parameters.

Generalized unitarity for the triple cut reads,

$$N|_{3-cut} + \frac{\Delta_{\text{box}}}{D_3}|_{3-cut} = (-i)^3\times$$
$$\sum_{h_1,h_2,h_3=\pm} A_1(-l^{*,-h_1}, p_1, (l^*-p_1)^{h_2})A_2((p_1-l^*)^{-h_2}, p_2, p_3, (l^*+p_4)^{h_3}) \times A_3((-l^*-p_4)^{-h_3}, p_4, l^{h_1}) \tag{53}$$

Note that the triple cut also detects the box information. Hence the left side must induce Δ_{box}/D_3. Since Δ_{box} is already calculated, we just need to subtract this term (OPP subtraction).

For the solution (I), we split the loop momenta in a spinors table,

$$\begin{array}{c|c|c} l & x1 & \tilde{4} \\ \hline l-p_1 & 1 & x\tilde{4}-\tilde{1} \\ \hline l+p_4 & x1+4 & \tilde{4} \end{array} \tag{54}$$

From the spinor table, we see that A_1 must be $\overline{\text{MHV}}$, and A_3 must be MHV. Here $x = a_3\langle 43\rangle/\langle 13\rangle$.

For the solution (II), we see the spinor table is

$$\begin{array}{c|c|c} l & y4 & \tilde{1} \\ \hline l-p_1 & y4-1 & \tilde{1} \\ \hline l+p_4 & 4 & y\tilde{1}+\tilde{4} \end{array} \tag{55}$$

From the spinor table, we see that A_1, must be MHV, and A_3 must be $\overline{\text{MHV}}$. Here $y = a_4\langle 13\rangle/\langle 43\rangle$.

B. Gluon amplitude $A^{(1)}(1^-2^-3^+4^+)$, triple cut

Recalled that

$$c_{\text{box},0}^{--++} = istA(1^-2^-3^+4^+), \quad c_{\text{box},1}^{--++} = 0 \tag{56}$$

For the solution (II), there is only one helicity configuration.

$$(-i)^3 \times \sum_{\text{helicities}} A_1 A_2 A_4|_{\text{solution (I)}} = \frac{it}{a_3+1} A(1^-2^-3^+4^+)\,. \tag{57}$$

On the other hand,

$$\frac{\Delta_{\text{box}}}{D_3}|_{\text{solution (I)}} = \frac{it}{a_3+1} A(1^-2^-3^+4^+)\,. \tag{58}$$

After the OPP subtraction, we determine that

$$c_{\text{tri},0} = c_{\text{tri},4} = c_{\text{tri},5} = c_{\text{tri},6} = 0\,. \tag{59}$$

For the solution (II), there are two helicity configurations

$$\begin{aligned}&(-i)^3 \times \sum_{\text{helicities}} A_1 A_2 A_4|_{\text{solution (II)}} \\ &= \left(-\frac{it(a_4 s + a_4 t - s)^3}{s^3} - \frac{i{a_4}^4 t(s+t)^4}{s^3(a_4 s + a_4 t - s)} \right) A(1^-2^-3^+4^+)\,. \end{aligned}\tag{60}$$

This expression is not a polynomial in a_4 and does not fit into the OPP basis. However, after the OPP subtraction, it is,

$$\begin{aligned}&(-i)^3 \times \sum_{\text{helicities}} A_1 A_2 A_4|_{\text{solution (II)}} - \frac{\Delta_{\text{box}}}{D_3}|_{\text{solution (I)}} \\ &= \left(\frac{8i(s+t)}{s} x_3 + \frac{8i(s+t)^2}{s^2 t} x_3^2 + \frac{16i(s+t)^3}{s^3 t^2} x_3^3 \right) A(1^-2^-3^+4^+) \end{aligned}\tag{61}$$

So we determine that

$$\begin{aligned}&c_{\text{tri},0} = 0, \quad c_{\text{tri},1} = \frac{8i(s+t)}{s} A(1^-2^-3^+4^+), \\ &c_{\text{tri},2} = \frac{8i(s+t)^2}{s^2 t} A(1^-2^-3^+4^+) \quad c_{\text{tri},3} = \frac{16i(s+t)^3}{s^3 t^2} A(1^-2^-3^+4^+)\,. \end{aligned}\tag{62}$$

Note that the two solutions are consistent in the sense that $c_{\text{tri},0}$'s values agree. All non-zero terms are spurious, so they do not contribute to the physical scattering amplitude.

C. Gluon amplitude $A^{(1)}(1^-2^+3^-4^+)$, triple cut

Recalled that

$$c_{\text{box},0}^{--++} = i\frac{st(s^2+st+t^2)^2}{(s+t)^4} A(1^-2^+3^-4^+), \quad c_{\text{box},1}^{--++} = 2i\frac{s^2 t(2s^2+3st+2t^2)}{(s+t)^4} A(1^-2^+3^-4^+) \tag{63}$$

For the solution (II), there is only one helicity configuration.

$$(-i)^3 \times \sum_{\text{helicities}} A_1 A_2 A_4|_{\text{solution (I)}} = \frac{it}{a_3+1} A(1^-2^-3^+4^+)\,. \tag{64}$$

But the messy terms from the top diagram kicked in,

$$\frac{\Delta_{\text{box}}}{D_3}\Big|_{\text{solution (I)}} = \frac{it\left(-2a_3s^3t - 3a_3s^2t^2 - 2a_3st^3 + s^4 + 2s^3t + 3s^2t^2 + 2st^3 + t^4\right)}{(a_3+1)(s+t)^4} A(1^-2^+3^-4^+)\,. \tag{65}$$

After the OPP subtraction, we determine that

$$c_{\text{tri},0} = \frac{ist^2\left(2s^2 + 3st + 2t^2\right)}{(s+t)^4} A(1^-2^+3^-4^+), \quad c_{\text{tri},4} = c_{\text{tri},5} = c_{\text{tri},6} = 0\,. \tag{66}$$

For the solution (II), there are two helicity configurations. After the OPP subtraction, again we determine that

$$c_{\text{tri},0} = \frac{ist^2\left(2s^2 + 3st + 2t^2\right)}{(s+t)^4} A(1^-2^+3^-4^+) \tag{67}$$

as well as some nonzero spurious coefficients.

We comment that for the $N = 4$ super-Yang-Mills theory, the triangle, bubble coefficients must be zero.

IV. D-DIMENSIONAL OPP REDUCTION

Here we briefly review the D-dimensional OPP reduction. This is a rigorous treatment of the integrand reduction and generalized unitarity. However, after the full computation, we see that for the one-loop case, the rigorous D-dimensional unitarity just provides the rational terms in the amplitude.

For instance, consider D-dimensional four-point massless box,

$$D_1 = l^2, \quad D_2 = (l-p_1)^2, \quad D_3 = (l-p_1-p_2)^2, \quad D_4 = (l+p_4)^2. \tag{68}$$

It is useful to use the van Neerven-Vermaseren parametrization.

$$x_1 = l\cdot p_1, \quad x_2 = l\cdot p_2, \quad x_3 = l\cdot p_4, \quad x_4 = l\cdot\omega \tag{69}$$

We further decompose the D-dimension l to the $4D$ part and the $D-4$ part,

$$l = l^{[4]} + l^{\perp} \tag{70}$$

We define $(l^{\perp})^2 \equiv -\mu_{11}$. It is easy to see that μ_{11} appears only linearly in D_i's.

$$D_i = f_i(x_1, x_2, x_3, x_4; s, t) - \mu_{11} \tag{71}$$

It is clearly that x_1, x_2 and x_3 are reducible scalar products (RSPs). μ_{11} and x_4 are ISPs. Furthermore from the OPP process or the Groebner basis computation, we see that

$$st^2 + \mu_{11}(-4st - 4t^2) - 4sx_4^2 = \text{Combination of } D_i\text{'s} \tag{72}$$

Hence we can trade x_4^2 for mu_{11}. Considering the renormalization condition, the OPP basis is then

$$\delta_{\text{box}} = c_{\text{box},0} + c_{\text{box},1}\mu_{11} + c_{\text{box},2}\mu_{11}^2 + c_{\text{box},3}x_4 + c_{\text{box},4}x_4\mu_{11} \tag{73}$$

There are five coefficients to be determined.

It is clear that $c_{\text{box},3}$ and $c_{\text{box},4}$ are spurious coefficients. From the explicit computation of the integrals [3], we see that in the limit $\epsilon \to 0$

$$I_{\text{box}}[1] \sim O(\epsilon^{-2}) \tag{74}$$

$$I_{\text{box}}[\mu_{11}] \sim \mathcal{O}(\epsilon) \tag{75}$$

$$I_{\text{box}}[\mu_{11}^2] \sim -\frac{1}{6} + \mathcal{O}(\epsilon) \tag{76}$$

So we see that in the epsilon expansion $I_{\text{box}}[\mu_{11}]$ provides no contribution to the one-loop amplitude. The $I_{\text{box}}[\mu_{11}^2]$ is finite and simply a rational number in the ϵ expansion. Usually, $c_{\text{box},4}$ is finite in the ϵ expansion which means that $c_{\text{box},4}$ only contributes to so-called *rational term*.

All the coefficient $c_{\text{box},i}$, $i = 0, \ldots 4$ can be determined by D-dimensional tree amplitudes [4]. However, there is a further shortcut [3].

$$A^{(1),\text{gluon}} = A^{(1),N=4} - 4A^{(1),N=1} + A^{(1),\text{scalar}} \tag{77}$$

where the scalar is the color-adjoint scalar particle. The $N = 4$ and $N = 1$ amplitudes are cut-constructible and no D-dimensional generalized unitarity is needed. For $A^{(1),\text{scalar}}$ we combine the D dimensional tree amplitudes to get the rational terms.

For the automatic D-dimensional one-loop amplitudes computations, check the Softwares GoSam, NGluon, CutTools [2, 5, 6].

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